

PFAFFIAN CALABI-YAU THREEFOLDS AND MIRROR SYMMETRY

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ABSTRACT. The aim of this article is to report on recent progress in understanding mirror symmetry for some non-complete intersection Calabi-Yau threefolds. We first construct four new smooth Calabi-Yau threefolds with $h^{1,1} = 1$ of non-complete intersection type, whose existence was previously conjectured in [17]. We then compute the period integral of candidate mirror families for these Calabi-Yau threefolds and check that the Picard-Fuchs equations coincide with the expected Calabi-Yau equations listed in [16, 17]. Some of the mirror families turn out to have two maximally unipotent monodromy points.

1. INTRODUCTION

The aim of this article is to report on recent progress in understanding mirror symmetry for some non-complete intersection Calabi-Yau threefolds. Throughout this paper, we adopt the following definition.

Definition 1.1. *A d -dimensional Calabi-Yau variety X is a normal compact variety with at worst Gorenstein canonical singularities and with trivial dualizing sheaf $\omega_X \cong \mathcal{O}_X$ such that $H^i(X, \mathcal{O}_X) = 0$, ($i = 1, 2, \dots, d-1$).*

We mainly discuss 3-dimensional Calabi-Yau varieties (Calabi-Yau threefolds) in this paper but most of the content can be easily generalized to other dimensional Calabi-Yau varieties. Among Calabi-Yau threefolds, those with $h^{1,1} = 1$ dimensional Kähler moduli space have been attracting much attention since their expected mirror partners have $h^{2,1} = 1$ dimensional complex moduli space and hence one can work on them in detail. There are around 30 known examples of topologically different smooth Calabi-Yau threefolds with $h^{1,1} = 1$, most of which are complete intersections in toric varieties and homogeneous spaces. Although non-complete intersection Calabi-Yau threefolds are only partially explored, they are intriguing on their own and provide important testing grounds for mirror symmetry. We thus think that the Calabi-Yau threefolds and mirror phenomena we report in this paper are of interest and will hopefully be the first step toward the future investigations. This paper is clearly influenced by E. Rødland's work [12], and we owe a lot of arguments to it. We mention it here, and do not repeat it each

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time in the sequel. In the following, we give a brief overview of this paper.

Section 2 is mainly devoted to the study of pfaffian threefolds in weighted projective spaces. pfaffian Calabi-Yau threefolds in \mathbb{P}^6 was first studied by F. Tonoli in his thesis [15]. By replacing the ambient space \mathbb{P}^6 by weighted projective spaces, we obtain new smooth Calabi-Yau threefolds with $h^{1,1} = 1$ of low degree. We then determine their fundamental topological invariants, namely $\int_X H^3$, $\int_X c_2(X) \cdot H$ and $\int_X c_3(X)$, which determine the topological type of X when it is simply connected and $h^{1,1} = 1$ according to the celebrated Wall's classification theorem [18]. The main result of this section is the following. See Section 2 for the definition of X_i and Y_i .

Theorem 1.1. *X_5, X_7, X_{10} and X_{25} are smooth Calabi-Yau threefolds with the following topological invariants.*

X_i	$h^{1,1}$	$h^{1,2}$	$\int_{X_i} H^3$	$\int_{X_i} c_2(X_i) \cdot H$
X_5	1	51	5	38
X_7	1	61	7	46
X_{10}	1	59	10	52
X_{25}	1	51	25	70

The existence of Calabi-Yau threefolds with these geometric invariants was previously conjectured from the view point of classification of Calabi-Yau equations by C. van Enkevort and D. van Straten in [17].

In Section 3 and 4, we report on mirror symmetry for these Calabi-Yau threefolds. A pfaffian Calabi-Yau threefold X_{13} of degree 13 was constructed by F. Tonoli [15] and a candidate of a mirror family for X_{13} was proposed by J. Böhm from the view point of tropical geometry in [1]. We first review the tropical mirror construction and its relation to orbifolding, and then justify the conjectural mirror family for X_{13} by computing the Picard-Fuchs equation. After computing the conjectural $g = 0, 1$ BPS invariants $\{n_d^g\}_{d \in \mathbb{N}}$, we heuristically determine the number of the degree 1 rational curves in a general degree 13 pfaffian Calabi-Yau threefold X_{13} and see it coincides with n_1^0 as the mirror symmetry predicts. Interestingly, the family has a special point where all the indices of the Picard-Fuchs equation are $1/2$, in addition to the usual maximally unipotent monodromy point at 0.

Although the existence of mirror families for given Calabi-Yau threefolds is highly non-trivial, by following the lead of the degree 13 case, we explicitly exhibit mirror families for Calabi-Yau threefolds X_5, X_7 and X_{10} by orbifolding and check that their Picard-Fuchs equations coincide with the expected Calabi-Yau equations, which are supposed to be the quantum differential equations of the initial threefolds. However, a generic member of these one-parameter families is quite singular and we could not find any crepant resolution.

Section 5 studies a degree 9 pfaffian Calabi-Yau threefold $X_9 \subset \mathbb{P}(1^6, 2)$, which is isomorphic to some \mathbb{P}_{32}^5 . However, this twofold interpretation allows X_9 to yield non-isomorphic special one-parameter families, both of which have the same Picard-Fuchs equation.

After we posted the preliminary version of this paper to the arXiv, a physics paper [13] studied open mirror symmetry of pfaffian Calabi-Yau threefolds constructed in this paper. It would be also interesting to extend Hori-Tong GLSM description [9] to our pfaffian threefolds.

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2. PFAFFIAN CALABI-YAU THREEFOLDS

2.1. Pfaffian Threefolds in Projective Spaces. Suppose that R is a regular local ring and $I \subset R$ is an ideal of height 1 or 2, then J. -P. Serre proved that R/I is Gorenstein if and only if it is complete intersection. This is no longer true for height 3 ideals, but such Gorenstein ideals are characterized as pfaffian ideals of certain skew-symmetric matrices [4]. This reasonably motivates us to study pfaffian varieties in the hope that they are not too general non-complete intersection varieties to investigate. In this subsection, we review the basics of pfaffians in order to make this paper self-contained. Although a pfaffian variety is, in general, a non-complete intersection, it has an explicit resolution of the structure sheaf, called the pfaffian resolution. The pfaffian resolution turns out to be indispensable tools to analyze the pfaffian variety.

In this paper we work over complex numbers \mathbb{C} . Let $\text{SkewSym}(n, \mathbb{C})$ be the set of $n \times n$ skew symmetric matrices. For $N = (n_{i,j}) \in \text{SkewSym}(n, \mathbb{C})$ the pfaffian $\text{Pf}(N)$ is defined as

$$\text{Pf}(N) = \frac{1}{r!2^r} \sum_{\sigma \in \mathfrak{S}_{2r}} \text{sign}(\sigma) \prod_{i=1}^r n_{\sigma(i)\sigma(r+i)}$$

if $n = 2r$ is even, and $\text{Pf}(N) = 0$ if n is odd. It is easy to see that $\text{Pf}(N)^2 = \det(N)$. We define N_{i_1, \dots, i_l} as a skew-symmetric matrix obtained by removing all the i_j -th rows and columns from N , and P_{i_1, \dots, i_l} as $\text{Pf}(N_{i_1, \dots, i_l})$. Set $n = 2r + 1$ from now on. The adjoint matrix of N is the matrix of rank 1, of the form

$$\text{adj}(N) = P \cdot P^t, \quad P = (P_1, P_2, \dots, P_{2r+1})^t.$$

We thus have $P \cdot N = \det(N) \cdot E_n = 0$ and if $\text{rank}(N) = 2r$ then $\{P_i\}_{i=1}^{2r+1}$ generate $\text{Ker}(N)$. $\text{GL}(n, \mathbb{C})$ acts on $\text{SkewSym}(n, \mathbb{C})$ by conjugation with a finite number of orbits $\{O_{2i}\}_{i=0}^r$ where the orbit O_{2i} consists of all skew-symmetric matrices of rank $2i$. The closure $\overline{O_{2i}}$ is singular along its boundary $\overline{O_{2i}} \setminus O_{2i}$ which consists of the union of $\{O_{2j}\}_{j=0}^{i-1}$.

Let us first recall the construction of pfaffian varieties in \mathbb{P}^n ($n > 3$). We identify $H^2(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$ via the hyperplane class. Given a locally free sheaf \mathcal{E} of odd rank $2r+1$ on \mathbb{P}^n and $t \in \mathbb{Z}$, a global section $N \in H^0(\mathbb{P}^n, \wedge^2 \mathcal{E}(t))$ defines an alternating morphism $\mathcal{E}^\vee(-t) \xrightarrow{N} \mathcal{E}$. Then the pfaffian complex associated to the data (\mathcal{E}, N) is the following complex

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-2s) \xrightarrow{P^t} \mathcal{E}^\vee(-t-s) \xrightarrow{N} \mathcal{E}(-s) \xrightarrow{P} \mathcal{O}_{\mathbb{P}^n},$$

where $s = c_1(\mathcal{E}) + rt$ and P is defined as;

$$P = \frac{1}{r!} \wedge^r N \in H^0(\mathbb{P}^n, \wedge^{2r} \mathcal{E}(rt)).$$

The first and third morphisms are given by taking the wedge product with P and P^t respectively. Note that, once we fix a basis of sections $\{e_i\}_{i=1}^{2r+1}$ of \mathcal{E} , N is just a matrix and we have

$$P = \sum_{i=1}^{2r+1} \text{Pf}(N_i) \bigwedge_{j \neq i} e_j.$$

Definition 2.1. A projective variety $X \subset \mathbb{P}^n$ is called a pfaffian variety associated to the data (\mathcal{E}, N) if the structure sheaf \mathcal{O}_X is given by $\text{Coker}(P)$. The sheaf $\text{Im}(P) \subset \mathcal{O}_{\mathbb{P}^n}$ is called the pfaffian ideal sheaf and denoted by \mathcal{I}_X .

When we do not specify the choice of N , it is understood that N is generically taken. In [15], F. Tonoli took advantage of the following globalized version of the classical theorem of D. A. Buchsbaum and D. Eisenbud and successfully gave some examples of smooth Calabi-Yau threefolds with $h^{1,1} = 1$ in \mathbb{P}^6 .

Theorem 2.1 (D. A. Buchsbaum, D. Eisenbud [4]). Let $X \subset \mathbb{P}^n$ be a pfaffian variety associated to the data (\mathcal{E}, N) . Then X is the degeneracy locus of the skew-symmetric map N and if N is generically of rank $2r$ it degenerates to rank $2r-2$ in the expected codimension 3, in which case, the pfaffian complex gives the self-dual resolution of the ideal sheaf of X . Moreover, X is locally Gorenstein, subcanonical with $\omega_X \cong \mathcal{O}_X(t+2s-n-1)$.

Let X be a pfaffian Calabi-Yau variety of dimension 3 in \mathbb{P}^6 . Here we have $t+2s=7$. By applying suitable twists, we can assume that $s=3$ and henceforth we consider the pfaffian complex of the following type.

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^6}(-7) \xrightarrow{P^t} \mathcal{E}^\vee(-4) \xrightarrow{N} \mathcal{E}(-3) \xrightarrow{P} \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

It is natural to expect some bounding of topological invariants of pfaffian Calabi-Yau threefolds in \mathbb{P}^6 . To see the restriction of the possible degree, we reduce the question to the compact complex surface theory by taking a hyperplane section. Let S be a compact, smooth complex surface. There are two important numerical invariants of S , namely a geometric genus $p_g = \dim H^0(S, K_S)$ and the self-intersection of the canonical divisor K_S^2 .

Theorem 2.2 (Castelnuovo inequality). *Let S be a minimal surface of general type. If the canonical map $\Phi_{|K_S|} : S \rightarrow \mathbb{P}^n$ is birational to the image, then $K_S^2 \geq 3p_g - 7$.*

We say a variety $X \subset \mathbb{P}^n$ is full if X is not contained in any hyperplane. Let S be a smooth surface obtained by taking a hyperplane section of a full Calabi-Yau threefold $X \subset \mathbb{P}^6$. Then $\deg(X) = K_S^2$ and K_S is nef since S is a canonical surface. As $X \subset \mathbb{P}^6$ is full, the short exact sequence $0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(1) \rightarrow \mathcal{O}_S(1) \rightarrow 0$ gives the dimension count $p_g = 6$ and thus the lower bound of the degree of X is 11. Since \mathbb{P}_{32}^5 is of degree 9, we cannot remove the fullness on X .

In [15], F. Tonoli constructed pfaffian Calabi-Yau threefolds of degree d from 11 to 17. Although Castelnuovo inequality tells us the minimal degree d of a full Calabi-Yau threefold in \mathbb{P}^6 is 11, there seems no smooth one with $d = 11$, and the case $d = 12$ turns out to be just a complete intersection $\mathbb{P}_{22,3}^6$. Therefore the degree 13 is a good starting point to analyze.

Definition 2.2 (F. Tonoli [15]). *The degree 13 pfaffian Calabi-Yau threefold $X_{13} \subset \mathbb{P}^6$ is the degeneracy locus of a generic alternating morphism $\mathcal{E}^\vee(-1) \xrightarrow{N} \mathcal{E}$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^6}(1) \oplus \mathcal{O}_{\mathbb{P}^6}^{\oplus 4}$.*

X_{13} is known to be smooth and the geometric invariants are computed to be

$$h^{1,1} = 1, \quad h^{1,2} = 61, \quad \int_{X_{13}} H^3 = 13, \quad \int_{X_{13}} c_2(X_{13}) \cdot H = 58.$$

The degree 14 pfaffian Calabi-Yau threefold X_{14} is defined by the locally free sheaf $\mathcal{E} = \mathcal{O}_{\mathbb{P}^6}^{\oplus 7}$. This is nothing but the intersection of \mathbb{P}^6 with $\text{Pfaff}(7) \subset \mathbb{P}^{20}$, the rank 4 locus of projectivised general skew-symmetric 7×7 matrices

$$\mathbb{P}\left(\bigwedge^2 \mathbb{C}^7\right) = \mathbb{P}(\text{SkewSym}(7, \mathbb{C})) \supset \text{Pfaff}(7) = \{[M] \mid \text{rank}(M) \leq 4\}.$$

X_{14} and its mirror partner are verified to have beautiful structures in [12, 3, 10]. We thus expect that the degree 13 pfaffian Calabi-Yau threefold X_{13} has as rich structures as X_{14} .

2.2. Pfaffian Threefolds in Weighted Projective Spaces. F. Tonoli's construction is generalized by replacing the ambient space \mathbb{P}^6 by any Fano variety. Special care must be taken when the ambient space is singular.

In the following we study the simplest case, when the ambient space is a weighted projective space $\mathbb{P}_{\mathbf{w}}$. Given a locally free sheaf \mathcal{E} of odd rank $2r + 1$ on $\mathbb{P}_{\mathbf{w}}$ and $t \in \mathbb{Z}$, a global section $N \in H^0(\mathbb{P}_{\mathbf{w}}, \wedge^2 \mathcal{E}(t))$ defines an alternating morphism $\mathcal{E}^\vee(-t) \xrightarrow{N} \mathcal{E}$. Then the pfaffian complex associated to the data (\mathcal{E}, N) is the following complex.

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_{\mathbf{w}}}(-t - 2s) \xrightarrow{P^t} \mathcal{E}^\vee(-t - s) \xrightarrow{N} \mathcal{E}(-s) \xrightarrow{P} \mathcal{O}_{\mathbb{P}_{\mathbf{w}}},$$

where $s = c_1(\mathcal{E}) + rt$ and $P = \frac{1}{r!} \wedge^r N$ are the same as before. The pfaffian variety X in $\mathbb{P}_{\mathbf{w}}$ associated to the data (\mathcal{E}, N) is a variety whose structure sheaf \mathcal{O}_X is given by $\text{Coker}(P)$. We also define $|\mathbf{w}|$ as a sum of weight of $\mathbb{P}_{\mathbf{w}}$.

Proposition 2.1. *Let $\mathbb{P}_{\mathbf{w}}$ be a weighted projective space of dimension 6 and (\mathcal{E}, N) data as above. Moreover, X has trivial dualizing sheaf $\omega_X \cong \mathcal{O}_X$ if and only if $t + 2s = |\mathbf{w}|$.*

Proof. Apply the functor $\mathcal{H}om(-, \omega_{\mathbb{P}_{\mathbf{w}}})$ to the pfaffian resolution to compute the dualizing sheaf $\omega_X \cong \mathcal{E}xt^3(\mathcal{O}_X, \omega_{\mathbb{P}_{\mathbf{w}}})$, which is isomorphic to $\cong \mathcal{O}_X$ if and only if $t + 2s = |\mathbf{w}|$ by the definition of the pfaffian variety. \square

As we are interested in Calabi-Yau threefolds, we restrict ourselves to the case $t + 2s = |\mathbf{w}|$. Moreover, up to an opportune twist, we may assume $t = 1$ for $|\mathbf{w}|$ odd and $t = 0$ for $|\mathbf{w}|$ even.

Definition 2.3. *Define X_5 , X_7 and X_{10} as pfaffian varieties associated to the following $\mathbb{P}_{\mathbf{w}_i}$, \mathcal{E}_i , and generic choices of $N \in H^0(\mathbb{P}_{\mathbf{w}_i}, \wedge^2 \mathcal{E}_i(t))$.*

i	\mathbf{w}_i	\mathcal{E}_i
5	$(1^4, 2^3)$	$\mathcal{O}_{\mathbb{P}_{\mathbf{w}_5}}^{\oplus 5}(1)$
7	$(1^5, 2^2)$	$\mathcal{O}_{\mathbb{P}_{\mathbf{w}_7}}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}_{\mathbf{w}_7}}^{\oplus 3}$
10	$(1^6, 2^1)$	$\mathcal{O}_{\mathbb{P}_{\mathbf{w}_{10}}}(1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}_{\mathbf{w}_{10}}}$

There are many other choices of weights \mathbf{w} and locally sheaves \mathcal{E} on $\mathbb{P}_{\mathbf{w}}$ to produce pfaffian Calabi-Yau threefolds but it seems only these three cases yield smooth Calabi-Yau threefolds in weighted projective spaces of dimension 6.

Theorem 2.3. *For a generic choice of N , the pfaffian varieties X_5 , X_7 and X_{10} are smooth varieties.*

Proof. A generic choice of N guarantees quasi-smoothness of X_i as follows. For X_5 $\text{Sing}(\mathbb{P}_{\mathbf{w}_5}) \cong \mathbb{P}^2$ and $X_5 \cap \text{Sing}(\mathbb{P}_{\mathbf{w}_5})$ is identified with the intersection of \mathbb{P}^2 with the rank 2 locus of projectivised general skew-symmetric 5×5 matrices, $\text{Pfaff}(5) \cap \mathbb{P}^2$, which is empty. For X_7 on $\text{Sing}(\mathbb{P}_{\mathbf{w}_7}) \cong \mathbb{P}^1$, the

defining matrix N has the following form

$$N = \begin{pmatrix} 0 & 0 & g_1 & g_2 & g_3 \\ 0 & 0 & g_4 & g_5 & g_6 \\ -g_1 & -g_4 & 0 & 0 & 0 \\ -g_2 & -g_5 & 0 & 0 & 0 \\ -g_3 & -g_6 & 0 & 0 & 0 \end{pmatrix},$$

where $\{g_i\}_{i=1}^6$ are linear polynomials of x_5 and x_6 , and has rank greater than 2 for a generic $\{g_i\}_{i=1}^6$. Finally for X_{10} generically $P_5|_{\text{Sing}(\mathbb{P}_{\mathbf{w}_{10}})} \neq 0$ while $P_i|_{\text{Sing}(\mathbb{P}_{\mathbf{w}_{10}})} = 0$ for $1 \leq i \leq 4$. So we henceforth assume that X_i avoids the singular locus $\text{Sing}(\mathbb{P}_{\mathbf{w}_i})$.

Let $\mathbb{P}_{\mathbf{w}_i}^{sm} = \mathbb{P}_{\mathbf{w}_i} \setminus \text{Sing}(\mathbb{P}_{\mathbf{w}_i})$. $H^0(\mathbb{P}_{\mathbf{w}_i}^{sm}, \wedge^2 \mathcal{E}_i(t))$ is generated by global sections and the surjection

$$H^0(\mathbb{P}_{\mathbf{w}_i}^{sm}, \wedge^2 \mathcal{E}_i(t)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}_{\mathbf{w}_i}^{sm}} \longrightarrow \wedge^2 \mathcal{E}_i(t)$$

induces a morphism f of $\mathbb{P}_{\mathbf{w}_i}^{sm}$ -schemes of full rank everywhere

$$\begin{array}{ccc} \mathbb{P}_{\mathbf{w}_i}^{sm} \times H^0(\mathbb{P}_{\mathbf{w}_i}^{sm}, \wedge^2 \mathcal{E}_i(t)) & \xrightarrow{f} & E = \text{Spec}(\text{Sym}(\wedge^2 \mathcal{E}_i(t))) \\ & \searrow \pi_1 \quad \swarrow \pi_2 & \\ & \mathbb{P}_{\mathbf{w}_i}^{sm} & \end{array}$$

sending $(x, N) \mapsto N(x)$. Let $p : \mathbb{P}_{\mathbf{w}_i}^{sm} \times H^0(\mathbb{P}_{\mathbf{w}_i}^{sm}, \wedge^2 \mathcal{E}_i(t)) \rightarrow H^0(\mathbb{P}_{\mathbf{w}_i}^{sm}, \wedge^2 \mathcal{E}_i(t))$ be the second projection. Define E_2 to be the codimension 3 variety of E whose fiber over a point $x \in \mathbb{P}_{\mathbf{w}_i}^{sm}$ is

$$O_2 \subset \text{SkewSym}(5, \mathbb{C}) \cong \pi_2^{-1}(x)$$

Note that O_2 is independent of the identification $\text{SkewSym}(5, \mathbb{C}) \cong \pi_2^{-1}(x)$. Then $Y = f^{-1}(E_2)$ is of codimension 3 and singular along $f^{-1}(\text{Sing}(E_2)) = \mathbb{P}_{\mathbf{w}_i}^{sm} \times \{0\}$. $p|_{Y \setminus (\text{Sing}(Y))}$ is dominant and the generic smoothness of $p|_{Y \setminus (\text{Sing}(Y))}$ proves that for a generic choice of $N \in H^0(\mathbb{P}_{\mathbf{w}_i}^{sm}, \wedge^2 \mathcal{E}_i(t))$

$$p|_{Y \setminus (\text{Sing}(Y))}^{-1}(N) = \{(x, N) \mid \text{rank}(N(x)) = 2\}$$

is smooth and of dimension 3. The quasi-smoothness of X_i then shows

$$\mathbb{P}_{\mathbf{w}_i}^{sm} \supset X_i = \pi_2(p|_{Y \setminus (\text{Sing}(Y))}^{-1}(N)) \cong p|_{Y \setminus (\text{Sing}(Y))}^{-1}(N).$$

□

For each X_i , vanishing of $H^j(X_i, \mathcal{O}_{X_i}) = 0$ for $j = 1, 2$ readily follows from the pfaffian resolution. Therefore X_5, X_7 and X_{10} are smooth Calabi-Yau threefolds. In the following, we assign to each X_i a polarization by the ample generator of $\text{Pic}(X_i)$, $\mathcal{O}_{X_i}(H)$ coming from its ambient space $\mathbb{P}_{\mathbf{w}_i}$.

Lemma 2.1. *The Hilbert series $H_{X_i}(t)$ for the pfaffian Calabi-Yau threefold X_i $i = 5, 7, 10$ are given by the following;*

$$H_{X_5}(t) = \frac{1 + 3t^2 + t^4}{(1-t)^4}, \quad H_{X_7}(t) = \frac{1 + t + 3t^2 + t^3 + t^4}{(1-t)^4},$$

$$H_{X_{10}}(t) = \frac{1 + 2t + 4t^2 + 2t^3 + t^4}{(1-t)^4}.$$

Proof. As we already have a resolution of the structure sheaf of X_i , the claim easily follows from the additivity of the Hilbert series and the formula

$$H_{\mathbb{P}_{\mathbf{w}_i}}(\mathcal{O}_{\mathbb{P}_{\mathbf{w}_i}}(k))(t) = \frac{t^k}{\prod_{i=0}^n (1 - t_i^{w_i})}.$$

□

Proposition 2.2. *The degree $\int_{X_i} H^3$ of the pfaffian Calabi-Yau threefold X_i is i .*

Proof. Since pfaffian variety X_i is locally a complete intersection, the triple intersection $\int_{X_i} H^3$ coincides with d , where d is $3!$ times the leading coefficient of the Hilbert polynomial $P_{X_i}(t)$, which is readily available thanks to the previous lemma. □

Proposition 2.3. *$\int_{X_i} c_2(X_i) \cdot H$ for each pfaffian Calabi-Yau 3-fold X_i is*

X_i	X_5	X_7	X_{10}
$\int_{X_i} c_2(X_i) \cdot H$	38	46	52

Proof. Since we know that X_i is a smooth Calabi-Yau threefold, Hirzebruch-Riemann-Roch Theorem gives

$$\chi(X_i, \mathcal{O}_{X_i}(H)) = \frac{1}{6} \deg(X_i) + \frac{1}{12} \int_{X_i} c_2(X_i) \cdot H.$$

By the Kodaira vanishing theorem, $H^j(X_i, \mathcal{O}_{X_i}(H)) = 0$ except for $j = 0$ and hence we have

$$\chi(X_i, \mathcal{O}_{X_i}(H)) = \dim H^0(X_i, \mathcal{O}_{X_i}(H)) = \dim H^0(\mathbb{P}_{\mathbf{w}_i}, \mathcal{O}_{\mathbb{P}_{\mathbf{w}_i}}(H))$$

This determine $\int_{X_i} c_2(X_i) \cdot H$. □

We will also need the resolutions of the powers of pfaffian ideals, which are intensively studied in [2]. Let R be a commutative ring. We consider a free R -module E of rank $2r + 1$ and a generic alternating map $N : E^\vee \rightarrow E$, then we have the pfaffian resolution.

$$0 \longrightarrow R \xrightarrow{P^t} E^\vee \xrightarrow{N} E \xrightarrow{P} R \longrightarrow R/I \longrightarrow 0.$$

Lemma 2.2 ([2]). *The resolution of I^2 has the following form.*

$$0 \longrightarrow L_{(2r-1)} E \cong \Lambda^{2r-1} E \xrightarrow{\vartheta_3} L_{(2r,1)} E \xrightarrow{\vartheta_2} L_{(2r+1,1^2)} E \cong S_2 E \xrightarrow{\vartheta_1} I^2 \longrightarrow 0,$$

where ϑ_1 is the second symmetric power of P and ϑ_3 and ϑ_2 are induced by the map

$$\Lambda^a E \otimes_S S_b E \rightarrow \Lambda^{a+1} E \otimes_S S_{b+1} E, \quad u \otimes v \mapsto \sum_{i,j} n_{i,j} e_i \wedge u \otimes v e_j,$$

where $N = \sum_{i,j} n_{i,j} e_i \otimes e_j$ with respect to some fixed basis for E .

Lemma 2.3. *The resolutions of the ideal sheaf $\mathcal{I}_{X_5}^2$, $\mathcal{I}_{X_7}^2$ and $\mathcal{I}_{X_{10}}^2$ are of the following form.*

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbb{P}_{\mathbf{w}_5}}(-12)^{\oplus 10} &\xrightarrow{\vartheta_3} \mathcal{O}_{\mathbb{P}_{\mathbf{w}_5}}(-10)^{\oplus 24} \xrightarrow{\vartheta_2} \mathcal{O}_{\mathbb{P}_{\mathbf{w}_5}}(-8)^{\oplus 15} \xrightarrow{\vartheta_1} \mathcal{I}_{X_5}^2 \longrightarrow 0 \\ 0 \longrightarrow \mathcal{O}_{\mathbb{P}_{\mathbf{w}_7}}(-12) \oplus \mathcal{O}_{\mathbb{P}_{\mathbf{w}_7}}(-11)^{\oplus 6} \oplus \mathcal{O}_{\mathbb{P}_{\mathbf{w}_7}}(-10)^{\oplus 3} \\ &\xrightarrow{\vartheta_3} \mathcal{O}_{\mathbb{P}_{\mathbf{w}_7}}(-10)^{\oplus 6} \oplus \mathcal{O}_{\mathbb{P}_{\mathbf{w}_7}}(-9)^{\oplus 12} \oplus \mathcal{O}_{\mathbb{P}_{\mathbf{w}_7}}(-8)^{\oplus 6} \\ &\xrightarrow{\vartheta_2} \mathcal{O}_{\mathbb{P}_{\mathbf{w}_7}}(-8)^{\oplus 6} \oplus \mathcal{O}_{\mathbb{P}_{\mathbf{w}_7}}(-7)^{\oplus 6} \oplus \mathcal{O}_{\mathbb{P}_{\mathbf{w}_7}}(-6)^{\oplus 3} \xrightarrow{\vartheta_1} \mathcal{I}_{X_7}^2 \longrightarrow 0 \\ 0 \longrightarrow \mathcal{O}_{\mathbb{P}_{\mathbf{w}_{10}}}(-10)^{\oplus 6} \oplus \mathcal{O}_{\mathbb{P}_{\mathbf{w}_{10}}}(-9)^{\oplus 4} &\xrightarrow{\vartheta_3} \mathcal{O}_{\mathbb{P}_{\mathbf{w}_{10}}}(-9)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}_{\mathbf{w}_{10}}}(-8)^{\oplus 16} \oplus \mathcal{O}_{\mathbb{P}_{\mathbf{w}_{10}}}(-7)^{\oplus 4} \\ &\xrightarrow{\vartheta_2} \mathcal{O}_{\mathbb{P}_{\mathbf{w}_{10}}}(-8) \oplus \mathcal{O}_{\mathbb{P}_{\mathbf{w}_{10}}}(-7)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}_{\mathbf{w}_{10}}}(-6)^{\oplus 10} \xrightarrow{\vartheta_1} \mathcal{I}_{X_{10}}^2 \longrightarrow 0 \end{aligned}$$

Here each term from left to right is regarded as 5×5 skew-symmetric, general but top left being zero, and symmetric matrices and the morphisms are given by $\vartheta_3(X) = NX - (NX)_{1,1}I$, $\vartheta_2(X) = XN + (XN)^t$, $\vartheta_1(X) = P^tXP$.

Proof. By suitably identify $\Lambda^3 F$, $L_{(4,1)}F$ and S_2F with respectively 5×5 skew-symmetric, general but top left being zero, and symmetric matrices, the morphisms ϑ_i are the above forms. \square

Theorem 2.4. *The Hodge numbers $h^{1,1}$ and $h^{1,2}$ of the pfaffian Calabi-Yau threefold X_i are given by the following table.*

X_i	X_5	X_7	X_{10}
$h^{1,1}$	1	1	1
$h^{1,2}$	51	61	59

Proof. In the following, we simply write $X = X_i$ and $\mathbb{P}_{\mathbf{w}} = \mathbb{P}_{\mathbf{w}_i}$ for some $i = 5, 7, 10$. Twisting the pfaffian resolution of the structure sheaf, we know that $H^i(X, \mathcal{O}_X(-j)) \cong H^{i+3}(\mathbb{P}_{\mathbf{w}}, \mathcal{O}_{\mathbb{P}_{\mathbf{w}}}(-|\mathbf{w}| - j))$ ($j = 1, 2$), which do not vanish only for $i = 3$. Restricting the weighted projective analogue of the Euler sequence on X , we obtain $0 \rightarrow \Omega_{\mathbb{P}_{\mathbf{w}}} \otimes_{\mathcal{O}_{\mathbb{P}_{\mathbf{w}}}} \mathcal{O}_X \rightarrow \bigoplus_{i=0}^6 \mathcal{O}_X(-w_i) \rightarrow \mathcal{O}_X \rightarrow 0$. Since we know that X is a smooth Calabi-Yau threefold, the long exact sequence of this gives, $H^i(X, \Omega_{\mathbb{P}_{\mathbf{w}}} \otimes_{\mathcal{O}_{\mathbb{P}_{\mathbf{w}}}} \mathcal{O}_X) = 0$ ($i = 0, 2$), $H^1(X, \Omega_{\mathbb{P}_{\mathbf{w}}} \otimes_{\mathcal{O}_{\mathbb{P}_{\mathbf{w}}}} \mathcal{O}_X) \cong \mathbb{C}$ and

$$\begin{aligned} 0 \longrightarrow H^2(X, \mathcal{O}_X) &\longrightarrow H^3(X, \Omega_{\mathbb{P}_{\mathbf{w}}} \otimes_{\mathcal{O}_{\mathbb{P}_{\mathbf{w}}}} \mathcal{O}_X) \\ &\longrightarrow H^3(X, \bigoplus_{i=0}^6 \mathcal{O}_X(-w_i)) \longrightarrow H^3(X, \mathcal{O}_X) \longrightarrow 0. \end{aligned}$$

Hence, we have $h^3(X, \Omega_{\mathbb{P}_{\mathbf{w}}} \otimes_{\mathcal{O}_{\mathbb{P}_{\mathbf{w}}}} \mathcal{O}_X) = h^3(X, \bigoplus_{i=0}^6 \mathcal{O}_X(-w_i)) - 1$. From the resolution of $\mathcal{F}_{\bullet} \rightarrow \mathcal{I}_X^2$ in Lemma 2.3 we obtain

$$h^4(\mathbb{P}_{\mathbf{w}}, \mathcal{I}_X^2) - h^5(\mathbb{P}_{\mathbf{w}}, \mathcal{I}_X^2) = \sum_{i=1}^3 (-1)^{i+1} h^6(\mathbb{P}_{\mathbf{w}}, \mathcal{F}_i) - h^6(\mathbb{P}_{\mathbf{w}}, \mathcal{I}_X^2).$$

The pfaffian resolution gives $H^4(\mathbb{P}_{\mathbf{w}}, \mathcal{I}_X) \cong H^6(\mathbb{P}_{\mathbf{w}}, \mathcal{O}_{\mathbb{P}_{\mathbf{w}}}(-|\mathbf{w}|)) \cong \mathbb{C}$ and $H^i(X, \mathcal{I}_X) = 0$ (otherwise). Since we know that X is indeed smooth, we have the following short exact sequence

$$0 \longrightarrow \mathcal{I}_X^2 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{N}_{X/\mathbb{P}_{\mathbf{w}}}^{\vee} \longrightarrow 0.$$

The induced long exact sequence gives $H^i(\mathbb{P}_{\mathbf{w}}, \mathcal{N}_{X/\mathbb{P}_{\mathbf{w}}}^{\vee}) = 0$ ($0 \leq i \leq 2$) and $H^5(\mathbb{P}_{\mathbf{w}}, \mathcal{I}_X^2) = H^6(\mathbb{P}_{\mathbf{w}}, \mathcal{I}_X^2) = 0$. Moreover, we also have the short exact sequence

$$0 \longrightarrow H^3(\mathbb{P}_{\mathbf{w}}, \mathcal{N}_{X/\mathbb{P}_{\mathbf{w}}}^{\vee}) \longrightarrow H^4(\mathbb{P}_{\mathbf{w}}, \mathcal{I}_X^2) \longrightarrow H^4(\mathbb{P}_{\mathbf{w}}, \mathcal{I}_X) \longrightarrow 0.$$

and dimension counting shows

$$h^3(\mathbb{P}_{\mathbf{w}}, \mathcal{N}_{X/\mathbb{P}_{\mathbf{w}}}^{\vee}) = h^3(X, \mathcal{N}_{X/\mathbb{P}_{\mathbf{w}}}^{\vee}) = \sum_{i=1}^3 (-1)^{i+1} h^6(\mathbb{P}_{\mathbf{w}}, \mathcal{F}_i).$$

On the other hand, the conormal exact sequence yields

$$0 \longrightarrow H^2(X, \Omega_X) \longrightarrow H^3(X, \mathcal{N}_{X/\mathbb{P}_{\mathbf{w}}}^{\vee}) \longrightarrow H^3(X, \Omega_{\mathbb{P}_{\mathbf{w}}} \otimes_{\mathcal{O}_{\mathbb{P}_{\mathbf{w}}}} \mathcal{O}_X) \longrightarrow 0$$

and $H^1(X, \Omega_X) \cong \mathbb{C}$. We establish the formula

$$\begin{aligned} h^2(X, \Omega_X) &= h^3(X, \mathcal{N}_{X/\mathbb{P}_{\mathbf{w}}}^{\vee}) - h^3(\Omega_{\mathbb{P}_{\mathbf{w}}} \otimes_{\mathcal{O}_{\mathbb{P}_{\mathbf{w}}}} \mathcal{O}_{X_{10}}) \\ &= \sum_{i=1}^3 (-1)^{i+1} h^6(\mathbb{P}_{\mathbf{w}}, \mathcal{F}_i) - h^3(X, \bigoplus_{i=0}^6 \mathcal{O}_X(-w_i)). \end{aligned}$$

Therefore $h^{1,2}$ is determined by the explicit description of $\mathcal{F}_{\bullet} \rightarrow \mathcal{I}_X^2$ derived in Lemma 2.3. \square

The existence of smooth Calabi-Yau threefolds X_5 , X_7 and X_{10} with the computed topological invariants was previously conjectured from the view point of Calabi-Yau equations in [17]. Regrettably it has not been settled yet whether they are simply connected or not.

2.3. Complete Intersection Type. In this subsection we study complete intersection of pfaffian varieties and hypersurfaces in weighted projective spaces. The idea is that we use pfaffian varieties as codimension 3 analogue of hypersurfaces in the ambient space.

Definition 2.4. Let $\mathcal{E}_{25} = \mathcal{O}_{\mathbb{P}^9}^{\oplus 5}$ a locally free sheaf on \mathbb{P}^9 . Two generic global sections $N_1, N_2 \in H^0(\mathbb{P}^9, \wedge^2 \mathcal{E}_{25}(1))$ define alternating morphisms $N_1, N_2 : \mathcal{E}_{25}^{\vee}(-t) \rightarrow \mathcal{E}_{25}$. Define X_{25} as common degeneracy loci of N_1 and N_2 .

Since the pfaffian sixfold associated to the data (\mathcal{E}_{25}, N_i) is $\text{Gr}(2, 5) \subset \mathbb{P}^9$, X_{25} is a complete intersection of two grassmannians embedded in two different ways $i_j : \text{Gr}(2, 5) \hookrightarrow \mathbb{P}^9$ ($j = 1, 2$).

$$X_{25} = i_1(\text{Gr}(2, 5)) \cap i_2(\text{Gr}(2, 5))$$

Lemma 2.4. *Let X be a pfaffian variety associated to (\mathcal{E}_{25}, N_i) . Then \mathcal{I}_X^2 has the following resolution.*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^9}(-6)^{\oplus 10} \longrightarrow \mathcal{O}_{\mathbb{P}^9}(-5)^{\oplus 24} \longrightarrow \mathcal{O}_{\mathbb{P}^9}(-4)^{\oplus 15} \longrightarrow \mathcal{I}_X^2 \longrightarrow 0$$

Proof. See Lemma 2.2. □

Proposition 2.4. *X_{25} is a smooth Calabi-Yau threefold with the following topological invariants.*

$$h^{1,1} = 1, \quad h^{1,2} = 51, \quad \int_{X_{25}} H^3 = 25, \quad \int_{X_{25}} c_2(X_{25}) \cdot H = 70$$

Proof. The basic strategy is to divide the construction of X_{25} into two steps and repeat the similar arguments in the previous subsection. The grassmannian description guarantees the smoothness of X_{25} . That X_{25} is a Calabi-Yau threefold and the determination of $\int_{X_{25}} H^3$ and $\int_{X_{25}} c_2(X_{25}) \cdot H$ follow immediately in the same manner as before. The only non-trivial part is the determination of the Hodge numbers and we sketch the proof.

Let X be a pfaffian sixfold associated to (\mathcal{E}_{25}, N_1) , which is $\text{Gr}(2, 5)$. Then $Y = X_{25}$ is a pfaffian threefold associated to $(\mathcal{O}_X^{\oplus 5}, N_2)$. A straightforward computation with the above lemma shows that $h^3(Y, \mathcal{N}_{Y/X}^\vee) = 75$ and there is an exact sequence

$$\begin{aligned} 0 \longrightarrow H^2(X, \Omega_X \otimes_{\mathcal{O}_{\mathbb{P}^w}} \mathcal{O}_Y) &\longrightarrow H^3(X, \mathcal{N}_{Y/X}^\vee \otimes_{\mathcal{O}_{\mathbb{P}^w}} \mathcal{O}_Y) \\ &\longrightarrow H^3(X, \Omega_{\mathbb{P}^9} \otimes_{\mathcal{O}_{\mathbb{P}^w}} \mathcal{O}_Y) \longrightarrow H^3(X, \Omega_X \otimes_{\mathcal{O}_{\mathbb{P}^w}} \mathcal{O}_Y) \longrightarrow 0. \end{aligned}$$

Combining this with the long exact sequence induced from the conormal sequence, we have

$$\begin{aligned} h^2(Y, \Omega_Y) &= h^3(Y, \mathcal{N}_{Y/X}^\vee) + h^2(X, \Omega_X \otimes_{\mathcal{O}_{\mathbb{P}^w}} \mathcal{O}_Y) - h^3(X, \Omega_X \otimes_{\mathcal{O}_{\mathbb{P}^w}} \mathcal{O}_Y) \\ &= 2h^3(Y, \mathcal{N}_{Y/X}^\vee) - h^3(X, \Omega_{\mathbb{P}^9} \otimes_{\mathcal{O}_{\mathbb{P}^w}} \mathcal{O}_Y) = 51. \end{aligned}$$

□

The existence of a smooth Calabi-Yau threefold with the computed topological invariants was already predicted in [17]. This Calabi-Yau equation has two maximally unipotent monodromy points of the same type, and this is clearly explained by the self-duality of $\text{Gr}(2, 5)$.

Example 2.1. *A complete intersection of a pfaffian variety associated with $\mathcal{F}_{10} = \mathcal{O}_{\mathbb{P}_{(17,2)}}^{\oplus 5}$ and a quartic hypersurface in $\mathbb{P}_{(17,2)}$ yields a smooth Calabi-Yau threefold Y_{10} with the following topological invariants.*

$$h^{1,1} = 1, \quad h^{1,2} = 101, \quad \int_{Y_{10}} H^3 = 10, \quad \int_{Y_{10}} c_2(Y_{10}) \cdot H = 64$$

We expect this to coincide with the double covering of Fano threefold in the list of C. Borcea [17].

Example 2.2. *A complete intersection of a pfaffian variety associated with $\mathcal{F}_5 = \mathcal{O}_{\mathbb{P}_{(16,2,3)}}^{\oplus 5}$ and a sextic hypersurface in $\mathbb{P}_{(16,2,3)}$ yields a singular Calabi-Yau threefold Y_5 . Interestingly, assuming smoothness of this threefold allow us to compute its topological invariants as follows*

$$h^{1,1} = 1, \quad h^{1,2} = 156, \quad \int_{Y_5} H^3 = 5, \quad \int_{Y_5} c_2(Y_5) \cdot H = 62.$$

The existence of a smooth Calabi-Yau threefold with the above invariants was predicted in [17].

The author is grateful to Makoto Miura for indicating the existence of Y_5 and Y_{10} . There are many other choices for locally free sheaves \mathcal{E} of odd rank and weights \mathbf{w} that yield Calabi-Yau threefolds, but there do not seem to exist any other smooth examples that are not previously known. There is, nevertheless, an interesting example X_9 , which we will deal with in Section 5.

3. MIRROR SYMMETRY FOR DEGREE 13 PFAFFIAN

3.1. Mirror Partner. Our main aim of this subsection is to explicitly construct a mirror family of X_{13} . As X_{13} is not a complete intersection Calabi-Yau threefold, the Batyrev-Borisov mirror construction is not applicable. Now we briefly review the tropical mirror construction proposed by J. Böhm. His construction reproduces the conventional Batyrev-Borisov mirror construction for complete intersection Calabi-Yau in toric Fano varieties. For a thorough treatment of the tropical mirror construction, we refer the reader to the original paper [1].

Let us start by recalling the Batyrev-Borisov mirror construction, following the usual notations in [6]. Let M and $N = \text{Hom}(M, \mathbb{Z})$ dual free abelian groups of rank d , and $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ be the scalar extension of M and N respectively. Suppose that \mathbb{P}_{Δ} is an n -dimensional toric variety associated to the normal fan Σ_{Δ} of an integral polytope $\Delta \subset M$. The Cox ring $S = \mathbb{C}[x_r | r \in \Sigma_{\Delta}(1)]$ of \mathbb{P}_{Δ} is graded by Chow group $A_{n-1}(\mathbb{P}_{\Delta})$ via the presentation sequence

$$0 \longrightarrow M \xrightarrow{A} \mathbb{Z}^{\Sigma_{\Delta}(1)} \longrightarrow A_{n-1}(\mathbb{P}_{\Delta}) \longrightarrow 0.$$

Suppose that Δ is reflexive and given a nef-partition $\Delta = \Delta_1 + \cdots + \Delta_k$ or equivalently $\Sigma_\Delta(1) = I_1 \cup \cdots \cup I_k$, then a complete intersection Calabi-Yau variety $X = V(I) \subset \mathbb{P}_\Delta$ of dimension $d = n - k$ is the zero locus of a generic section $(f_i)_{i=1}^k \in H^0(\mathbb{P}_\Delta, \bigoplus_{i=1}^k \mathcal{O}_{\mathbb{P}_\Delta}(E_i))$, where $\bigotimes_{i=1}^k \mathcal{O}_{\mathbb{P}_\Delta}(E_i) \cong -K_{\mathbb{P}_\Delta}$ corresponds to the nef-partition.

Define $\nabla_i = \text{Conv.}(\{0\} \cup I_i)$ and the Minkowski sum $\nabla = \nabla_1 + \cdots + \nabla_k \subset N$. Then the following holds.

$$\Delta^* = \text{Conv.}(\nabla_1 \cup \cdots \cup \nabla_k), \quad \nabla^* = \text{Conv.}(\Delta_1 \cup \cdots \cup \Delta_k)$$

$\nabla = \nabla_1 + \cdots + \nabla_k$ is again reflexive and this gives a nef-partition of ∇ , called the dual nef-partition. We define a complete intersection Calabi-Yau variety \check{X} by using $\nabla \subset N$. Choosing a maximal projective subdivision of the normal fan of Δ and ∇ , we get families \mathcal{X} and $\check{\mathcal{X}}$ of Calabi-Yau varieties, which are conjectured to form a mirror pair. The important point is that giving a nef-partition is essentially equivalent to determining a union of toric varieties $X_0 = V(I_0)$ to which fibers of \mathcal{X} maximally degenerates.

Let I_0 be a reduced monomial ideal in the Cox ring S . The degree 0 homomorphisms $\text{Hom}(I_0, S/I_0)_0$ form a finite dimensional vector space. The torus $T = \mathbb{C}^{\Sigma_\Delta(1)}$ acts on S and thus on $\text{Hom}(I_0, S/I_0)_0$ as well. So the vector space has a basis of deformations which are characters of T . Being of degree 0, any such character ρ corresponds to an element $m_\rho \in M \cong \text{Im}(A)$. Given a flat family \mathcal{X} of Calabi-Yau varieties in \mathbb{P}_Δ with special fiber X_0 such that the corresponding ideal $I_0 \subset S$ is a reduced monomial ideal. We represent the complex moduli space of a generic fiber X of \mathcal{X} by a one-parameter family \mathcal{X}' ; Take a T -invariant basis $\rho_1, \dots, \rho_l \in \text{Hom}(I_0, S/I_0)_0$ of the tangent space of the component of Hilbert scheme containing \mathcal{X} at X_0 and assume that the tangent vector $v = \sum_i^l a_i \rho_i$ of \mathcal{X}' at X_0 satisfies $a_i \neq 0 \forall i$. The elements ρ_1, \dots, ρ_l correspond to elements $m_1, \dots, m_l \in M$ of the lattice of monomials of \mathbb{P}_Δ . The construction of the first order deformation of a mirror family $\check{\mathcal{X}}$ comes with a natural map via the interpretation of lattice points as deformations and divisors, cf. the monomial divisor map [6]. Take the convex hull ∇^* of m_1, \dots, m_l and define \mathbb{P}_∇ the toric variety associated to the normal fan of the (not necessarily integral) polytope ∇ . Then the toric divisors of \mathbb{P}_∇ and the induced divisors on a prospective mirror inside will correspond to deformations of X_0 in \mathcal{X} . The Bergman complex of X_0 defines a special fiber $\check{X}_0 \subset \mathbb{P}_\nabla$ and the first order deformations $\check{\mathcal{X}}$ of contributing to the mirror degeneration \check{X}_0 are constructed by the lattice points of the support of $\text{Strata}(X_0)^* \subset \Delta^*$. It is sufficient to know given family up to first order deformation in the case of complete intersection or pfaffian varieties as their deformations are not obstructed.

To relate $\check{\mathcal{X}}$ to the initial family \mathcal{X} . We need to blow-down the ambient toric variety \mathbb{P}_∇ to an orbifold quotient of a weighted projective space \mathbb{P}_w/G ,

contracting all divisors which do not correspond to Fermat deformation of \mathcal{X} . This blow-down is in general not unique and we choose appropriate one on case-by-case basis. The next one-parameter family was conjectured to be the mirror family of the degree 13 pfaffian Calabi-Yau threefold X_{13} . Note that this family has a special monomial fiber \check{X}_0 on $t = 0$ and the first order deformation of \check{X}_0 is given by the Fermat deformation.

Definition 3.1 (J. Böhm [1]). *Define $\check{\mathcal{X}} = \{\check{X}_t\}_{t \in \mathbb{P}^1}$ as the one-parameter flat family of the pfaffian Calabi-Yau threefolds associated to the following special skew-symmetric 5×5 matrix N_t parametrized by $t \in \mathbb{P}^1$.*

$$N_t = \begin{pmatrix} 0 & tx_0^2 & x_5x_6 & x_3x_4 & tx_2^2 \\ -tx_0^2 & 0 & t(x_3 + x_4) & x_2 & x_1 \\ -x_5x_6 & -t(x_3 + x_4) & 0 & tx_1 & x_0 \\ -x_3x_4 & -x_2 & -tx_1 & 0 & t(x_5 + x_6) \\ -tx_2^2 & -x_1 & -x_0 & -t(x_5 + x_6) & 0 \end{pmatrix}$$

Of course, $\check{\mathcal{X}}$ is nothing but a special one-parameter family of degree 13 pfaffian Calabi-Yau threefolds. More explicitly, the pfaffian ideal sheaf of this family $\mathcal{I}_{\check{\mathcal{X}}} = \langle P_1, P_2, P_3, P_4, P_5 \rangle$ is generated by

$$\begin{aligned} P_1 &= x_0x_2 - tx_1^2 - t^2(x_3 + x_4)(x_5 + x_6) \\ P_2 &= x_0x_3x_4 - tx_5x_6(x_5 + x_6) - t^2x_1x_2^2 \\ P_3 &= x_1x_3x_4 - tx_2^3 - t^2x_0^2(x_5 + x_6) \\ P_4 &= x_1x_5x_6 - tx_0^3 - t^2x_2^2(x_3 + x_4) \\ P_5 &= x_2x_5x_6 - tx_3x_4(x_3 + x_4) - t^2x_0^2x_1. \end{aligned}$$

Since \check{X}_t is originally contained in the toric variety $\mathbb{P}^6/\mathbb{Z}_{13}$, \mathbb{Z}_{13} acts on \check{X}_t as

$$\zeta_{13} \cdot [x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] = [x_0 : \zeta_{13}^4 x_1 : \zeta_{13}^8 x_2 : \zeta_{13}^{10} x_3 : \zeta_{13}^{10} x_4 : \zeta_{13}^{11} x_5 : \zeta_{13}^{11} x_6],$$

where $\zeta_{13} = e^{\frac{2\pi i}{13}}$. The fixed points of \check{X}_t under this action are 6 points, which do not depend on the value of parameter t . Among them, there are 4 singular points $p_i = \{x_i \neq 0, x_j = 0 \ (j \neq i)\}$ ($i = 3, 4, 5, 6$), and 2 smooth ones $p_{i,i+1} = \{x_i + x_{i+1} = 0, x_i \neq 0, x_j = 0 \ (j \neq i, i+1)\}$ ($i = 3, 5$).

Proposition 3.1. *For generic $t \in \mathbb{P}^1$, singular points of \check{X}_t appear only on $\{p_i\}_{i=3}^6$ and each has multiplicity 12.*

Proof. We work on the singular point p_3 , for example. In a neighborhood of p_3 , since $P_{1,4,5} \neq 0$, \check{X}_t is defined by the complete intersection of P_1 , P_4 and P_5 . Then it is easily seen that the germ of this singularity is isomorphic to the following cDV singularity

$$f = x^2 + y^3 + z^5 + zw^2, \ (x, y, z, w) \in \mathbb{C}^4.$$

Here the action of \mathbb{Z}_{13} is given by $\zeta_{13} \cdot (x, y, z, w) = (\zeta_{13}^{11}x, \zeta_{13}^3y, \zeta_{13}^7z, \zeta_{13}w)$. The Milnor number of this singularity turns out to be 12. On the other

hand, the Jacobian ideal of $\mathcal{J}_{\check{X}}$ has dimension 0 and degree 48¹. Due to the symmetry, other singular points are of multiplicity 12 as well and hence we conclude the singular points are only $\{p_i\}_{i=3}^6$. \square

Now we have a family of Calabi-Yau threefolds $\mathcal{X} = \{\check{X}_t\}_{t \in \mathbb{P}^1}$ parametrized by $t \in \mathbb{P}^1$. However, this is not an effective family, as $\check{X}_t \cong \check{X}_{\zeta_7 t}$ for $\zeta_7 = e^{\frac{2\pi i}{7}}$ via the map

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \mapsto [x_0 : \zeta_7^3 x_1 : x_2, \zeta_7^6 x_3 : \zeta_7^6 x_4 : \zeta_7^6 x_5 : \zeta_7^6 x_6].$$

Remark 3.1. *It is proved in [11] that the above cD_6 singularity admits no crepant resolution. However, the quotient $\{f = 0\}/\mathbb{Z}_{13}$ may have a crepant resolution. The definition of this family will be partly justified by calculating the Picard-Fuchs equation of it in the next subsection.*

3.2. Period Map and Picard-Fuchs Equation. Since X_{13} is a smooth Calabi-Yau threefold, it has a nowhere vanishing holomorphic 3-form up to multiplication with a non-zero constant. Although a pfaffian variety is not defined as a complete intersection and there is no way of explicitly getting one in general, there is an analogous way of obtaining a global section of $\omega_{X_{13}} \cong \Omega_{X_{13}}^3$. For the sake of simplicity, we restrict ourselves to the degree 13 case, but the generalization to the case of general pfaffian Calabi-Yau threefolds is straightforward.

Proposition 3.2 (E. Rødland [12]). *Let $\sigma \in \mathfrak{S}_5$. We have a nowhere vanishing global section of $\Omega_{X_{13}}^3 \cong \mathcal{O}_{\mathbb{P}^6}(-7) \otimes \bigwedge^3 \mathcal{N}_{X_{13}/\mathbb{P}^6}$, up to multiplication with a non-zero constant, explicitly given by*

$$\alpha = C_\sigma \text{Res}_X \frac{P_{\sigma(1), \sigma(2), \sigma(3)} \Omega_0}{P_{\sigma(1)} P_{\sigma(2)} P_{\sigma(3)}},$$

where

$$\Omega_0 = \frac{1}{(2\pi i)^6} \sum_{i=0}^6 (-1)^i x_i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_6.$$

This expression is independent of the choice of σ so long as the constant C_σ is chosen appropriately.

Proof. First of all, the invariance of the integrand under scaling of the coordinates can be checked for each σ and thus it is well-defined as a rational 6-form on \mathbb{P}^6 . On the affine open $U_{\sigma(4), \sigma(5)} = \{P_{\sigma(1), \sigma(2), \sigma(3)} \neq 0\}$, $\{P_{\sigma(i)}\}_{i=1}^3$ form a complete intersection and thus $\{P_{\sigma(i)}\}_{i=1}^3$ can be seen as a part of the local coordinate. We may assume that $\{P_{\sigma(1)}, P_{\sigma(2)}, P_{\sigma(3)}, x_4, x_5, x_6, x_7\}$ form the coordinate of \mathbb{A}^7 , i.e. $\frac{\partial(P_{\sigma(1)}, P_{\sigma(2)}, P_{\sigma(3)})}{\partial(x_1, x_2, x_3)} \neq 0$. Since we have

$$dP_{\sigma(1)} \wedge dP_{\sigma(2)} \wedge dP_{\sigma(3)} = \sum_{i < j < k} \frac{\partial(P_{\sigma(1)}, P_{\sigma(2)}, P_{\sigma(3)})}{\partial(x_i, x_j, x_k)} dx_i \wedge dx_j \wedge dx_k,$$

¹This is done by Macaulay2.

the residue theorem provides the following holomorphic 3-form on $X_{13}|_{U_{\sigma(4),\sigma(5)}}$,

$$\alpha = C_\sigma \frac{P_{\sigma(1),\sigma(2),\sigma(3)}}{(2\pi i)^3 \frac{\partial(P_{\sigma(1)}, P_{\sigma(2)}, P_{\sigma(3)})}{\partial(x_0, x_1, x_2)}} \sum_{i=4}^7 (-1)^i x_i dx_4 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_7.$$

On the other hand, $P_{\sigma(1),\sigma(2),\sigma(3)}$ vanishes if and only if $\{P_{\sigma(i)}\}_{i=1}^3$ do not form a complete intersection. Therefore $\frac{\partial(P_{\sigma(1)}, P_{\sigma(2)}, P_{\sigma(3)})}{\partial(x_1, x_2, x_3)}$ divides $P_{\sigma(1),\sigma(2),\sigma(3)}$, and α is globally defined. Furthermore, the local expression of α shows it is nowhere vanishing on X_{13} . Since X_{13} is Calabi-Yau threefold, $\Omega_{X_{13}}^3$ is trivial and hence the expression of α for each ν is merely different by a constant. \square

Although \check{X}_t has some singularities, the nowhere vanishing holomorphic 3-form α is defined on the non-singular locus $\check{X}_t \setminus \text{Sing}(\check{X}_t)$ and the period map of the family $\check{\mathcal{X}}$ is defined as usual since integration can be performed on 3-cycle away from the singular locus. For the sake of convenience, we work with the variety \check{X}_t in \mathbb{P}^6 instead of $\check{X}_t/\mathbb{Z}_{13}$. Note that Picard-Fuchs equations is preserved under taking a quotient of the variety by a finite group under whose action α is invariant. More precisely, we can perform the integration on \check{X}_t and obtain the genuine period by dividing by 13.

At $t = 0$, the variety \check{X}_t decomposes into 13 3-dimensional planes and thus the origin is a good candidate for a maximally unipotent monodromy point of this one-parameter family $\check{\mathcal{X}}$. The fundamental period can be obtained by integrating the holomorphic 3-form on a torus cycle that vanishes at $t = 0$. Fix a 3-dimensional plane H defined by $H = \{x_1 = x_2 = x_3 = 0\}$. On the domain $H \setminus (\{x_4 = 0\} \cup \{x_5 = 0\} \cup \{x_6 = 0\})$, there is a cycle given by $|\frac{x_4}{x_0}| = |\frac{x_5}{x_0}| = |\frac{x_6}{x_0}| = \epsilon$, which extends to a 3-dimensional torus cycle $\gamma \in H_3(\check{X}_t, \mathbb{C})$ ($|t| \ll 1$).

Theorem 3.1. *Let $\Phi_0(t) = \int_\gamma \alpha$ be a period map of the one-parameter family $\{\check{X}_t\}_{t \in \mathbb{P}^1}$. Then $\Phi_0(t)$ has the following expansion near $t = 0$.*

$$\Phi_0(t) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 \sum_{k=0}^n \binom{2n+k}{n} \binom{n}{k}^2 t^{7n}$$

It can be observed that $\phi = t^7 \in \mathbb{P}^1$ is the genuine moduli parameter of $\{\check{X}_t\}_{t \in \mathbb{P}^1}$. Thus we shall write $\Phi_0(\phi)$ and $\{\check{X}_\phi\}_{\phi \in \mathbb{P}^1}$. Moreover, the Picard-Fuchs operator \mathcal{D} of this family is

$$\begin{aligned} \mathcal{D} = & 13^2 \Theta^4 - \phi(59397 \Theta^4 + 117546 \Theta^3 + 86827 \Theta^2 + 28054 \Theta + 3380) \\ & + 2^4 \phi^2 (6386 \Theta^4 - 1774 \Theta^3 - 17898 \Theta^2 - 11596 \Theta - 2119) \\ & + 2^8 \phi^3 (67 \Theta^4 + 1248 \Theta^3 + 1091 \Theta^2 + 312 \Theta + 26) - 2^{12} \phi^4 (2 \Theta + 1)^4, \end{aligned}$$

where Θ is the Euler operator $\phi \frac{\partial}{\partial \phi}$.

Proof. Let us work on an affine open subset $U_0 = \{x_0 = 1\}$ and fix the permutation $\nu = (2, 4, 5, 1, 3)$. For the sake of convenience, we define $a_{i,j}$ to be

$$(a_{i,j}) = \begin{pmatrix} \frac{x_5^2 x_6}{x_3 x_4} t & \frac{x_5 x_6^2}{x_3 x_4} t & \frac{x_1 x_5^2}{x_3 x_4} t^2 \\ \frac{1}{x_1 x_5 x_6} t & \frac{x_5^2 x_3}{x_1 x_5 x_6} t^2 & \frac{x_5^2 x_4}{x_1 x_5 x_6} t^2 \\ \frac{x_5^2 x_4}{x_2 x_5 x_6} t & \frac{x_3 x_4}{x_2 x_5 x_6} t & \frac{x_1}{x_2 x_5 x_6} t^2 \end{pmatrix}.$$

Then, near the origin, the period integral is described as

$$\Phi_0(t) = \int_{\gamma} \text{Res}_X \frac{P_{2,4,5}}{P_2 P_4 P_5} \Omega_0 = \int_{\Gamma} \frac{1}{\prod_{i=1,3,4} (1 - \sum_{j=1}^3 a_{i,j})} \cdot \bigwedge_{k=1}^6 \frac{dx_k}{2\pi i x_k},$$

where $\Gamma = \{|x_i| = \epsilon \ (i = 1, \dots, 6)\}$. We then expand the denominator of the integrand as a power series in terms of $a_{i,j}$ and see that the only terms that contribute the period integral is the products $\prod a_{i,j}^{n_{i,j}}$ that is independent of x_i . Suppose $\prod a_{i,j}^{n_{i,j}}$ ($n_{i,j} \in \mathbb{Z}_{\geq 0}$) does not contain any x_i , then $\prod a_{i,j}^{n_{i,j}}$ is a product of

$$\begin{aligned} t_1 &= a_{1,1} a_{1,2} a_{2,3} a_{3,1} a_{3,3} = t^7 = \phi \\ t_2 &= a_{1,1} a_{1,2} a_{2,2} a_{3,2} a_{3,3} = t^7 = \phi \\ t_3 &= a_{1,1} a_{1,2} a_{1,3} a_{2,1} a_{3,1} a_{3,2} = t^7 = \phi \end{aligned}$$

and it is easily checked that this expression is unique. Therefore the period integral $\Phi_0(t)$ is essentially a function of $\phi = t^7$, and henceforth we write $\Phi_0(\phi)$. Note that this is compatible with the previous observation that $\check{X}_t \cong \check{X}_{\zeta^7 t}$. Since

$$t_1^a t_2^b t_3^c = a_{1,1}^{a+b+c} a_{1,2}^{a+b+c} a_{1,3}^c a_{3,1}^c a_{3,2}^b a_{3,3}^a a_{4,1}^{a+c} a_{4,2}^{b+c} a_{4,3}^{a+c}$$

and the coefficient of $\prod a_{i,j}^{n_{i,j}}$ appearing as an integrand $\Phi_0(\phi)$ is $\prod_{i=1,3,4} \binom{n_{i,1}+n_{i,2}+n_{i,3}}{n_{i,1}, n_{i,2}, n_{i,3}}$, the period integral $\Phi_0(\phi)$ can be summarized as follows.

$$\begin{aligned} \Phi_0(\phi) &= \sum_{n=0}^{\infty} \sum_{a+b+c=n} \binom{2a+2b+3c}{a+b+c, a+b+c, c} \binom{a+b+c}{c, b, a} \binom{2a+2b+2c}{a+c, b+c, a+b} \phi^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{l=0}^{n-k} \binom{2n+k}{n} \binom{n+k}{n} \binom{n}{k} \binom{n-k}{l} \binom{2n}{n-l} \binom{n+l}{k+l} \phi^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{2n+k}{n} \binom{n+k}{n} \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} \binom{2n}{n+k} \binom{n+k}{n-l} \phi^n \\ &= \sum_{n=0}^{\infty} \binom{2n}{n} \sum_{k=0}^n \binom{n+k}{n} \binom{2n}{n} \sum_{l=0}^{n-k} \binom{n-k}{l} \binom{n+k}{n-l} \phi^n \\ &= \sum_{n=0}^{\infty} \binom{2n}{n}^2 \sum_{k=0}^n \binom{2n+k}{n} \binom{n}{k}^2 \phi^n, \end{aligned}$$

where we used relations

$$\begin{aligned} \binom{2n}{n-l} \binom{n+l}{k+l} &= \binom{2n}{n+k} \binom{n+k}{n-l}, \quad \binom{n+k}{n} \binom{2n}{n+k} = \binom{2n}{n} \binom{n}{k}, \\ \sum_{l=0}^{n-k} \binom{n-k}{l} \binom{n+k}{n-l} &= \binom{2n}{k}. \end{aligned}$$

$\Phi_0(\phi)$ coincides with the power series solution of the Calabi-Yau equation of No.99 classified in [16], whose Picard-Fuchs equation is exactly what we are looking for. \square

Corollary 3.1. *Let α_1, α_2 be the roots of $256\phi^2 + 349\phi - 1$. Then the Riemann's P-Scheme of \mathcal{D} is given by the following. The conifold points are α_1, α_2 .*

$$\left\{ \begin{array}{c|ccccc} \phi & 0 & \alpha_1 & \alpha_2 & 13/16 & \infty \\ \hline \rho_1 & 0 & 0 & 0 & 0 & 1/2 \\ \hline \rho_2 & 0 & 1 & 1 & 1 & 1/2 \\ \hline \rho_3 & 0 & 1 & 1 & 3 & 1/2 \\ \hline \rho_4 & 0 & 2 & 2 & 4 & 1/2 \end{array} \right\}$$

As we have expected, \mathcal{D} has a maximally unipotent monodromy point at $\phi = 0$ and other singular points of \mathcal{D} are on $\mathbb{R} \cup \infty$. Observe that ∞ is not a maximally unipotent monodromy point in the usual sense but very similar to that. This will be discussed later in this paper.

3.3. Picard-Fuchs Equation around 0 and Curve Counting. We now briefly review Gromov-Witten and BPS invariants. Let X be a Calabi-Yau threefold and let $N_\beta^g(X) = \int_{[\overline{M}_{g,0}(X,\beta)]^{vir}} 1$ be the 0-point genus g Gromov-Witten invariant of X in the curve class $\beta \in H_2(X, \mathbb{Z})$. Here $[\overline{M}_{g,0}(X, \beta)]^{vir}$ is the virtual fundamental class of the coarse moduli space of stable maps $\overline{M}_{g,0}(X, \beta)$ of expected complex dimension $(1-g)(\dim X - 3) + \int_\beta c_1(X) = 0$.

Definition 3.2. *Define BPS (Gopakumar-Vafa) invariants $n_\beta^g(X)$ by the formula*

$$\sum_{\beta \neq 0} \sum_{g \geq 0} N_\beta^g(X) \lambda^{2g-2} q^\beta = \sum_{\beta \neq 0} \sum_{g \geq 0} n_\beta^g(X) \sum_{k > 0} \frac{1}{k} (2 \sin(\frac{kt}{2}))^{2g-2} q^{k\beta}.$$

LHS is the generating function of Gromov-Witten invariants of $N_\beta^g(X)$ of X in all genera and all nonzero curve classes. Matching the coefficients of the two series yields equations determining $n_\beta^g(X)$ recursively in terms $N_\beta^g(X)$.

As the Picard-Fuchs operator \mathcal{D} of $\check{\mathcal{X}} = \{\check{X}_\phi\}_{\phi \in \mathbb{P}^1}$ has a maximally unipotent monodromy point at $\phi = 0$, we can define the mirror map $q(\phi)$ there and calculate the conjectural genus g BPS invariants $\{n_d^g\}_{d \in \mathbb{N}}$ of X_{13} . In what follows, we work on the case $g = 0, 1$ for simplicity. Since it is a routine work to calculate the mirror map and the Yukawa couplings, we omit the detail of those computations below. For a complete description,

see for example [5, 6].

A good integral basis of $H_3(\check{X}_\phi, \mathbb{Z})$, which corresponds to the normalized solutions of \mathcal{D} below, determines a canonical coordinate q of complexified Kähler moduli of X_{13} . At $\phi = 0$, we have two normalized solutions of \mathcal{D} , $\Phi_0(\phi)$ and $\Phi_1(\phi)$. $\Phi_0(\phi)$ is of the form of power series such that $\Phi_0(0)=1$ and $\Phi_1(\phi)$ satisfies

$$\Phi_1(\phi) = (\log(\phi))\Phi_0(\phi) + \Psi(\phi),$$

where $\Psi(\phi)$ is regular at $\phi = 0$ and $\Psi(0) = 0$. These give the mirror map $q(\phi) = \exp(\frac{\Phi_1(\phi)}{\Phi_0(\phi)})$.

$$q(\phi) = \phi + 86\phi^2 + 12901\phi^3 + 2460318\phi^4 + 536898026\phi^5 + \dots$$

The quantum corrected Yukawa coupling

$$K_{ttt}(q) = \int_X H^3 + (q \frac{d}{dq})^3 \sum_{d \geq 1} N_d(X_{13}) q^d \in \mathbb{Q}[[q]]$$

is then determined by the mirror map $q(\phi)$ to be

$$K_{ttt}(q) = 13 + 647q + 129975q^2 + 25451198q^3 + 5134100919q^4 + \dots$$

We will apply the following BCOV formula for $g = 1$ Gromov-Witten potential $F_1(\phi)$ to X_{13} .

$$F_1(\phi) = \frac{1}{2} \log \left\{ \frac{\Phi_0(\phi)^{\frac{\chi(X)}{12} - 3 - h^{1,1}} (q \frac{d\phi}{dq})}{disc(\phi)^{\frac{1}{6}} \phi^{\frac{\int_X c_2(X) \cdot H}{12} + 1}} \right\}$$

Here we assumed that the exponent of the discriminant is $1/6$ as usual. Then the $g = 0, 1$ BPS invariants are computed to be

d	n_d^0	n_d^1
1	647	0
2	16166	0
3	942613	176
4	80218296	164696
5	8418215008	78309518

Since we have the defining equations of X_{13} , we can in principle count the number of degree d rational curves on general X_{13} and check the coincidence of it with n_d^0 as follows. Write a map $\mathbb{P}^1 \rightarrow \mathbb{P}^6$ as

$$[u : v] \mapsto [\sum_{i=0}^d a_i u^i v^{d-i} : \sum_{i=0}^d b_i u^i v^{d-i} : \dots : \sum_{i=0}^d g_i u^i v^{d-i}].$$

Then the image of this map is contained in X_{13} if and only if $P_i(\mathbf{x}(u, v)) = 0$ ($i = 1, \dots, 5$) for all $[u : v] \in \mathbb{P}^1$. This containment condition yields dependent equations in $\{a_0, a_1, \dots, g_{d-1}, g_d\}$. Since what we want to count is not maps from \mathbb{P}^1 to X_{13} but rational curves in X_{13} , we must kill $\text{Aut}(\mathbb{P}^1)$ by suitably normalizing the map. As the proportional polynomials also define

the same map, the number of the independent parameters turns out to be $7(d+1) - 3 - 1$. We predict that the ideal generated by the dependent equations has dimension 0 and the degree n_d^0 .

When $d = 1$, we have 19 dependent equations and 10 parameters. For a generic choice of N and with suitable normalization, we actually compute the degree of the ideal and get the answer 647, as the mirror symmetry predicts². Explicit calculation is available upon request.

3.4. Picard-Fuchs equation around ∞ . In [12], E. Rødland constructed a mirror family for the degree 14 pfaffian Calabi-Yau threefolds $X_{14} = \text{Pfaff}(7) \cap \mathbb{P}^6$ by orbifolding the initial threefolds. His work is notable from two aspects. Firstly, this is the first example of mirror symmetry for a non-complete intersection Calabi-Yau threefold with $h^{1,1} = 1$. Secondly, the Picard-Fuchs equation of the mirror family \check{X}_{14} has two maximally degenerating points; ∞ corresponds to the initial X_{14} and 0 corresponds to $\text{Gr}(2, 7) \cap \mathbb{P}^{13} \subset \mathbb{P}^{20}$, which is the projective dual of $\text{Pfaff}(7) \cap \mathbb{P}^6 \subset \mathbb{P}^{20}$. In fact, this pair is the first example of a derived equivalence between non-birational Calabi-Yau threefolds [3]. K. Hori and D. Tong presented how to describe these Calabi-Yau threefolds with GLSM using a non-abelian gauge group in two dimensions [9]. The link between the pfaffian X_{14} and the grassmannian sections $\text{Gr}(2, 7)_{17}$ was further studied in [10], in which a thought-provoking phenomenon in the higher genus Gromov-Witten invariants is discovered.

In our case, ∞ is apparently an interesting point of \mathcal{D} and it seems worthwhile to analyze it in detail³. We will first see that it makes sense to call ∞ a maximally unipotent monodromy point and calculate the virtual invariants there. Changing the coordinate from ϕ to $1/\phi$ and transforming the gauge by $\sqrt{\phi}$ amount to the change, $\Theta \rightarrow -\Theta \rightarrow -\Theta - 1/2$, in the Euler operator. Let us also change the variable from ϕ to $-\phi$ for later use. Then the Picard-Fuchs operator becomes

$$\begin{aligned} \mathcal{D}' = & 2^{20}\Theta^4 - 2^8\phi(1072\Theta^4 - 17824\Theta^3 - 10888\Theta^2 - 1976\Theta - 145) \\ & + 2^5\phi^2(51088\Theta^4 + 116368\Theta^3 - 45264\Theta^2 - 14228\Theta - 1397) \\ & + 13\phi^3(73104\Theta^4 + 1536\Theta^3 - 488\Theta^2 + 384\Theta + 97) + 13^2\phi^4(2\Theta + 1)^4. \end{aligned}$$

Although \mathcal{D}' has a maximally unipotent monodromy point at $\phi = 0$, the integrality of mirror symmetry breaks. It is observed that there is a preferable choice of variable $\check{\phi} = \phi/2^{16}$, and then everything miraculously goes well. The coefficients $1/2^{16}$ is chosen to be the minimum, for which the integrality of the normalized period, the mirror map and the BPS invariants holds, as

²This is done by Macaulay2.

³This type of special point also appears when we consider a Calabi-Yau threefold \mathbb{P}_{24}^7 . The quantum differential equation of \mathbb{P}_{24}^7 is $\theta^4 - 2^4q(2\theta + 1)^4$.

we will see ⁴.

The Picard Fuchs operator $\tilde{\mathcal{D}}$ with respect to this new variable is

$$\begin{aligned}\tilde{\mathcal{D}} = & \tilde{\Theta}^4 - 2^4 \tilde{\phi} (1072 \tilde{\Theta}^4 - 17824 \tilde{\Theta}^3 - 10888 \tilde{\Theta}^2 - 1976 \tilde{\Theta} - 145) \\ & + 2^{17} \tilde{\phi}^2 (51088 \tilde{\Theta}^4 + 116368 \tilde{\Theta}^3 - 45264 \tilde{\Theta}^2 - 14228 \tilde{\Theta} - 1397) \\ & + 13 \cdot 2^{28} \tilde{\phi}^3 (73104 \tilde{\Theta}^4 + 1536 \tilde{\Theta}^3 - 488 \tilde{\Theta}^2 + 384 \tilde{\Theta} + 97) \\ & + 13^2 2^{44} \tilde{\phi}^4 (2 \tilde{\Theta} + 1)^4.\end{aligned}$$

This is the Calabi-Yau equation of No.225 in [17]. However, the expected positive Euler number corresponding to this point [16] excludes a geometric interpretation by a Calabi-Yau threefold with $h^{1,1} = 1$. Since $\tilde{\mathcal{D}}$ has a maximally unipotent monodromy point at 0, it still makes sense to speak of the virtual BPS invariants $\{\tilde{n}_d^0\}_{d \in \mathbb{N}}$ corresponding to 0,

d	\tilde{n}_d^0
1	70944a
2	107300032a
3	3707752060576a
4	66327758316665792a
5	1970671594871618215520a

where a is supposed to be the degree of the virtual geometry⁵. We hope to understand the meaning of these virtual invariants, which may not come from Calabi-Yau geometry.

3.5. Conclusion. It is classically known that the monodromy matrix of the quantum differential equation with respect to an appropriate basis is expressed in terms of the geometric invariants of the underlying Calabi-Yau threefold with one dimensional moduli. In what follows, we assume that the origin is a maximally unipotent monodromy point. Then $\int_X H^3$ and $\int_X c_2(X) \cdot H$ can be read off from the monodromy around the origin and the conifold point. After it is analytically continued to the origin, the conifold-period $z_2(t)$ has the form

$$z_2(t) = \frac{\int_X H^3}{6} t^3 + \frac{\int_X c_2(X) \cdot H}{24} t + \frac{\int_X c_3(X)}{(2\pi i)^3} \zeta(3) + \sum_{d=1}^{\infty} N_d^0(X) q^d,$$

where $q = e^{2\pi i t}$. So we obtain $\int_X c_3(X)$ as well and have consistency check of $\int_X c_2(X) \cdot H$. It was numerically verified in [17] that the invariants computed from the differential equation \mathcal{D} coincides with the fundamental geometric invariants of the degree 13 pfaffian Calabi-Yau threefold X , $\int_X H^3$, $\int_X c_2(X) \cdot H$ and $\int_X c_3(X)$. One feature of mirror symmetry is that the interchange of Hodge numbers, but as \check{X} admits no crepant resolution,

⁴S. Hosono pointed out that the change of the sign and the coefficient $1/2^{16}$ can be justified by the analytic continuation of the local solutions about 0 to ∞ .

⁵ a is expected to be 1 in [17].

our claim that (a suitable resolution of) $\check{X}_t/\mathbb{Z}_{13}$ is a mirror partner for X is based on the coincidence the fundamental geometric invariants mentioned above. An alternative and preferable way of the verification is direct computation of the Gromov-Witten invariants of X_{13} , such as [14].

Conjecture 1. *The BPS invariants of the degree 13 pfaffian Calabi-Yau threefold X_{13} coincides with the numbers $\{n_d^g\}_{d \in \mathbb{N}}$ we calculated above, as mirror symmetry predicts.*

4. MIRROR SYMMETRY FOR DEGREE 5, 7, 10 PFAFFIANS

4.1. Mirror Partners. Following the lead of the degree 13 case, we construct mirror families for the smooth Calabi-Yau threefolds we obtained in the section 2, except the degree 25 case. Since we do not know a systematic way of finding an appropriate blow-down, we omit the finding procedure of mirror partners in this paper. See Appendix for the conjectural BPS invariants computed via families of Calabi-Yau threefolds in this section.

Definition 4.1. *Define $\check{\mathcal{X}}_5 = \{\check{X}_{5,t}\}_{t \in \mathbb{P}^1}$ as the one-parameter family of degree 5 pfaffian Calabi-Yau threefolds $\check{X}_{5,t}$ associated to the following special skew-symmetric 5×5 matrix $N_{5,t}$ parametrized by $t \in \mathbb{P}^1$.*

$$N_{5,t} = \begin{pmatrix} 0 & tx_6 & x_4 & x_0x_1 & tx_5 \\ -tx_6 & 0 & t(x_0^2 + x_1^2) & x_5 & x_2x_3 \\ -x_4 & -t(x_0^2 + x_1^2) & 0 & t(x_2^2 + x_3^2) & x_6 \\ -x_0x_1 & -x_5 & -t(x_2^2 + x_3^2) & 0 & tx_4 \\ -tx_5 & -x_2x_3 & -x_6 & -tx_4 & 0 \end{pmatrix}$$

This is a conjectural mirror family for X_5 and degenerates to a union of toric varieties with normal crossings at $t = 0$. In fact we choose $\check{X}_{5,0}$ as a candidate for the fiber over a maximally unipotent monodromy point and deform it so that the first order deformation is Fermat type. Then the deformation automatically extends to higher orders, so long as it is a pfaffian Calabi-Yau threefold, and we obtain $\check{\mathcal{X}}_5$.

\mathbb{Z}_{10} acts on $\check{X}_{5,t}$ as follows

$$\zeta_{10} \cdot [x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] = [x_0 : x_1 : \zeta_{10}x_2 : \zeta_{10}x_3 : \zeta_{10}^4x_4 : \zeta_{10}^6x_5 : \zeta_{10}^3x_6],$$

where $\zeta_{10} = e^{\frac{2\pi i}{10}}$. There are 8 fixed points under this action, $p_{\pm i} = \{x_i \pm x_{i+1} = 0, x_i \neq 0, x_j = 0 \ (j \neq i, i+1)\}$ ($i = 0, 2$) which are singular with multiplicity 2. Each is locally isomorphic to the following cDV singularity

$$f_5 = x^3 + y^2 + zw = 0, \ (x, y, z, w) \in \mathbb{C}^4,$$

where the action of \mathbb{Z}_{10} is given by $\zeta_{10} \cdot (x, y, z, w) = (\zeta_{10}^4x, \zeta_{10}^6y, \zeta_{10}z, \zeta_{10}w)$. Since $\dim(\text{Sing}(\check{X}_{5,t})) = 0$ and $\deg(\text{Sing}(\check{X}_{5,t})) = 8$, there is no other singular point on $\check{X}_{5,t}$.

It is observed that $\check{\mathcal{X}}_5 = \{\check{X}_{5,t}\}_{t \in \mathbb{P}^1}$ is not an effective family, as $\check{X}_{5,t} \cong \check{X}_{5,\zeta_{10}t}$ for $\zeta_{10} = e^{\frac{2\pi i}{10}}$ via the map

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \mapsto [x_0 : x_1 : x_2 : \zeta_{10}^5 x_3 : \zeta_{10}^4 x_4 : \zeta_{10}^7 x_5 : \zeta_{10}^9 x_6].$$

Proposition 4.1. *The period integral of the mirror family $\{\check{X}_{5,t}\}_{t \in \mathbb{P}^1}$ is computed to be*

$$\Phi_0(\phi) = \sum_{n=0}^{\infty} \binom{2n}{n} \sum_k^n \binom{n}{k} \binom{n+k}{n} \binom{2n+2k}{n+k} \binom{2n+k}{2n-k} \phi^n$$

and the Picard-Fuchs operator \mathcal{D}_5 is given by the following.

$$\begin{aligned} \mathcal{D}_5 = & \Theta^4 - 2^2 \phi (500\Theta^4 + 976\Theta^3 + 677\Theta^2 + 189\Theta + 19) \\ & + 2^4 \phi^2 (3968\Theta^4 + 3968\Theta^3 - 1336\Theta^2 - 1164\Theta - 177) \\ & - 2^{10} \phi^3 (500\Theta^4 + 24\Theta^3 - 37\Theta^2 + 6\Theta + 3) + 2^{12} \phi^4 (2\Theta + 1)^4, \end{aligned}$$

where we put $\phi = t^{10}$.

Proof. The computation is almost identical to the degree 13 pfaffian case. \square

This Picard-Fuchs equation \mathcal{D}_5 is the Calabi-Yau equation of No.302 listed in [16]. The topological invariants computed from \mathcal{D}_5 coincide with those of X_5 as we expected.

Corollary 4.1. *Let α_1, α_2 be the roots of $256\phi^2 - 1968\phi + 1$, then Riemann's P-Scheme of \mathcal{D}_5 is given by the following.*

$$\left(\begin{array}{c|ccccc} \phi & 0 & \alpha_1 & \alpha_2 & 1/16 & \infty \\ \hline \rho_1 & 0 & 0 & 0 & 0 & 1/2 \\ \hline \rho_2 & 0 & 1 & 1 & 1 & 1/2 \\ \hline \rho_3 & 0 & 1 & 1 & 3 & 1/2 \\ \hline \rho_4 & 0 & 2 & 2 & 4 & 1/2 \end{array} \right)$$

Interestingly enough, the Picard-Fuchs operator \mathcal{D}_5 has two special points, 0 and ∞ . There is again a preferable new variable $\tilde{\phi} = \phi/2^8$ and the Picard-Fuchs equation around ∞ with respect to this new variable is identical to the initial one. Hence, both of the special points 0 and ∞ seem to correspond to the Calabi-Yau threefold X_5 in this case.

Definition 4.2. *Define $\check{\mathcal{X}}_7 = \{\check{X}_{7,t}\}_{t \in \mathbb{P}^1}$ as the one-parameter family of degree 7 pfaffian Calabi-Yau threefolds $\check{X}_{7,t}$ associated to the following special skew-symmetric 5×5 matrix $N_{7,t}$ parametrized by $t \in \mathbb{P}^1$.*

$$N_{7,t} = \begin{pmatrix} 0 & tx_2^3 & x_0x_1 & x_5 & tx_6 \\ -tx_2^3 & 0 & tx_5 & x_6 & x_3x_4 \\ -x_0x_1 & -tx_5 & 0 & t(x_3 + x_4) & x_2 \\ -x_5 & -x_6 & -t(x_3 + x_4) & 0 & t(x_0 + x_1) \\ -tx_6 & -x_3x_4 & -x_2 & -t(x_0 + x_1) & 0 \end{pmatrix}$$

\mathbb{Z}_7 acts on $\check{X}_{7,t}$ as

$$\zeta_7 \cdot [x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] = [x_0 : x_1 : \zeta_7^4 x_2 : \zeta_7 x_3 : \zeta_7 x_4 : \zeta_7^3 x_5 : \zeta_7^6 x_6],$$

where $\zeta_7 = e^{\frac{2\pi i}{7}}$. This action has 6 fixed points, which do not depend on the value of parameter t . We have $\dim(\text{Sing}(\check{X}_{7,t})) = 1$ and $\deg(\text{Sing}(\check{X}_{7,t})) = 4$. $\text{Sing}(\check{X}_{7,t})$ passes through 2 of the above fixed points, namely $p_{i,i+1} = \{x_i + x_{i+1} = 0, x_i \neq 0, x_j = 0 \ (j \neq i, i+1)\}$ ($i = 0, 3$).

It is observed that $\check{\mathcal{X}}_7 = \{\check{X}_{7,t}\}_{t \in \mathbb{P}^1}$ is not an effective family, as $\check{X}_{7,t} \cong \check{X}_{7,\zeta_9 t}$ for $\zeta_9 = e^{\frac{2\pi i}{9}}$ via the map

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \mapsto [x_0 : x_1 : \zeta_9^2 x_2 : x_3 : x_4 : \zeta_9^8 x_5 : \zeta_9^8 x_6].$$

Proposition 4.2. *The period integral of the mirror family $\{\check{X}_{7,t}\}_{t \in \mathbb{P}^1}$ is computed to be*

$$\Phi_0(\phi) = \sum_{n=0}^{\infty} \binom{2n}{n} \sum_{k=0}^{2n} \binom{n+k}{k} \binom{2n}{k}^2 \phi^n$$

and the Picard-Fuchs operator \mathcal{D}_7 is given by the following.

$$\begin{aligned} \mathcal{D}_7 = & 7^2 \Theta^4 - 2 \cdot 3 \cdot 7 \phi (1272 \Theta^4 + 2508 \Theta^3 + 1779 \Theta^2 + 525 \Theta + 56) \\ & + 2^2 3 \phi^2 (43704 \Theta^4 + 38088 \Theta^3 - 25757 \Theta^2 - 20608 \Theta - 3360) \\ & - 2^4 3^3 \phi^3 (2736 \Theta^4 - 1512 \Theta^3 - 1672 \Theta^2 - 357 \Theta - 14) \\ & - 2^6 3^5 \phi^4 (2 \Theta + 1)^2 (3 \Theta + 1) (3 \Theta + 2), \end{aligned}$$

where we put $\phi = t^9$.

Proof. The computation is almost identical to the degree 13 pfaffian case. \square

This Picard-Fuchs equation \mathcal{D}_7 is the Calabi-Yau equation of No.109 listed in [16]. The topological invariants computed from \mathcal{D}_7 coincide with those of X_7 as we expected.

Corollary 4.2. *Let α_1, α_2 be the roots of $432\phi^2 + 1080\phi - 1$, then Riemann's P-Scheme of \mathcal{D}_7 is given by the following.*

$$\left(\begin{array}{c|ccccc} \phi & 0 & \alpha_1 & \alpha_2 & 7/36 & \infty \\ \hline \rho_1 & 0 & 0 & 0 & 0 & 1/2 \\ \hline \rho_2 & 0 & 1 & 1 & 1 & 1/2 \\ \hline \rho_3 & 0 & 1 & 1 & 3 & 1/3 \\ \hline \rho_4 & 0 & 2 & 2 & 4 & 2/3 \end{array} \right)$$

0 is the only maximally unipotent monodromy point of \mathcal{D}_7 .

Definition 4.3. Define $\check{\mathcal{X}}_{10} = \{\check{X}_{10,t}\}_{t \in \mathbb{P}^1}$ as the one-parameter family of degree 10 pfaffian Calabi-Yau threefolds $\check{X}_{10,t}$ associated to the following special skew-symmetric 5×5 matrix $N_{10,t}$ parametrized by $t \in \mathbb{P}^1$.

$$N_{10,t} = \begin{pmatrix} 0 & tx_4^2 & x_0x_1 & x_6 & t(x_2 + x_3) \\ -tx_4^2 & 0 & tx_6 & x_2x_3 & x_5 \\ -x_0x_1 & -tx_6 & 0 & tx_5^2 & x_4 \\ -x_6 & -x_2x_3 & -tx_5^2 & 0 & t(x_0 + x_1) \\ -t(x_2 + x_3) & -x_5 & -x_4 & -t(x_0 + x_1) & 0 \end{pmatrix}$$

\mathbb{Z}_{10} acts on $\check{X}_{10,t}$ as

$$\zeta_{10} \cdot [x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] = [x_0 : x_1 : \zeta_{10}^6 x_2 : \zeta_{10}^6 x_3 : \zeta_{10}^9 x_4 : \zeta_{10}^7 x_5 : \zeta_{10}^1 x_6],$$

where $\zeta_{10} = e^{\frac{2\pi i}{10}}$. There are 6 fixed points, all of which are singular and do not depend on the value of parameter t . Among them, 4 points $p_i = \{x_i \neq 0, x_j = 0 \ (j \neq i)\}$ ($i = 0, 1, 2, 3$) appear with multiplicity 12 and other 2 $p_{i,i+1} = \{x_i + x_{i+1} = 0, x_i \neq 0, x_j = 0 \ (j \neq i, i+1)\}$ ($i = 0, 2$) with multiplicity 7. The singularities are locally isomorphic to the following cDV singularities, for $(x, y, z, w) \in \mathbb{C}^4$.

$$f_{10} = x^4 + y^2 + z^2w + zw^2 = 0,$$

where the action of \mathbb{Z}_{10} is given by $\zeta_{10} \cdot (x, y, z, w) = (\zeta_{10}^7 x, \zeta_{10}^9 y, \zeta_{10}^6 z, \zeta_{10}^6 w)$, and

$$g_{10} = x^8 + y^2 + zw = 0,$$

where $\zeta_{10} \cdot (x, y, z, w) = (\zeta_{10}^9 x, \zeta_{10} y, \zeta_{10}^6 z, \zeta_{10}^6 w)$. Since $\dim(\text{Sing}(\check{X}_{10})) = 0$ and $\deg(\text{Sing}(\check{X}_{10})) = 62$, there is no other singular point on $\check{X}_{10,t}$.

It is observed that $\check{\mathcal{X}}_{10} = \{\check{X}_{10,t}\}_{t \in \mathbb{P}^1}$ is not an effective family, as $\check{X}_{10,t} \cong X_{10, \zeta_{16}^2 t}$ for $\zeta_{16} = e^{\frac{2\pi i}{16}}$ via the map

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \mapsto [x_0 : x_1 : x_2 : x_3 : \zeta_{16}^3 x_4 : \zeta_{16}^3 x_5 : \zeta_{16}^7 x_6].$$

Proposition 4.3. The period integral of the mirror family $\{\check{X}_{10,t}\}_{t \in \mathbb{P}^1}$ is computed to be

$$\Phi_0(\phi) = \sum_{n=0}^{\infty} \binom{2n}{n} \sum_{k=0}^{2n} (-1)^{k+n} \binom{2n}{k}^4 \phi^n$$

and the Picard-Fuchs operator \mathcal{D}_{10} is given by the following.

$$\begin{aligned} \mathcal{D}_{10} = & 5^2 \Theta^4 - 2^2 5 \phi (688 \Theta^4 + 1352 \Theta^3 + 981 \Theta^2 + 305 \Theta + 35) \\ & + 2^4 \phi^2 (5856 \Theta^4 + 7008 \Theta^3 + 96 \Theta^2 - 1260 \Theta - 265) \\ & - 2^{10} \phi^3 (176 \Theta^4 + 120 \Theta^3 + 69 \Theta^2 + 30 \Theta + 5) + 2^{12} \phi^4 (2 \Theta + 1)^4, \end{aligned}$$

where we put $\phi = t^8$.

Proof. The computation is almost identical to the degree 13 pfaffian case. \square

This Picard-Fuchs equation \mathcal{D}_{10} is the Calabi-Yau equation of No.263 listed in [16]. The topological invariants computed from \mathcal{D}_{10} coincide with those of X_{10} as we expected.

Corollary 4.3. *Let α_1, α_2 be the roots of $256\phi^2 - 544\phi + 1$, then Riemann's P-Scheme of \mathcal{D}_{10} is*

$$\left\{ \begin{array}{c|ccccc} \phi & 0 & \alpha_1 & \alpha_2 & 5/16 & \infty \\ \hline \rho_1 & 0 & 0 & 0 & 0 & 1/2 \\ \hline \rho_2 & 0 & 1 & 1 & 1 & 1/2 \\ \hline \rho_3 & 0 & 1 & 1 & 3 & 1/2 \\ \hline \rho_4 & 0 & 2 & 2 & 4 & 1/2 \end{array} \right\}.$$

The Picard-Fuchs operator \mathcal{D}_{10} has two special points, 0 and ∞ . The Picard-Fuchs operator $\tilde{\mathcal{D}}_{10}$ around ∞ with respect to the new variable $\tilde{\phi} = 1/(\phi 2^{12})$ is

$$\begin{aligned} \tilde{\mathcal{D}}_{10} = & \tilde{\Theta}^4 - 2^4 \tilde{\phi} (704 \tilde{\Theta}^4 + 928 \tilde{\Theta}^3 + 612 \tilde{\Theta}^2 + 148 \tilde{\Theta} + 13) \\ & + 2^{12} \tilde{\phi}^2 (5856 \tilde{\Theta}^4 + 4704 \tilde{\Theta}^3 - 1632 \tilde{\Theta}^2 - 972 \tilde{\Theta} - 121) \\ & - 2^{20} 5 \tilde{\phi}^3 (2752 \tilde{\Theta}^4 + 96 \tilde{\Theta}^3 - 60 \tilde{\Theta}^2 + 24 \tilde{\Theta} + 7) + 2^{28} 5^2 \tilde{\phi}^4 (2 \tilde{\Theta} + 1)^4. \end{aligned}$$

This is the Calabi-Yau equation of No.271 listed in [16]. The existence of a Calabi-Yau threefold with the corresponding topological invariants was predicted in [17]. It would be interesting to find the Calabi-Yau threefold and study the relationship with X_{10} .

4.2. Conclusion. As a generic member of the mirror family is quite singular just as the degree 13 case and we don't know if it admits any crepant resolution, the verification of the mirror phenomena is again based on the coincidence of the fundamental geometric invariants of the initial Calabi-Yau threefold with the invariants obtained from monodromy calculation of the Picard-Fuchs equation of the mirror family [17].

Conjecture 2. *The BPS invariants of the pfaffian Calabi-Yau threefold X_i ($i = 5, 7, 10$) coincides with the numbers $\{n_d^g\}_{d \in \mathbb{N}}$ in Appendix as mirror symmetry predicts.*

5. ANOTHER EXAMPLE

Although we could not find any other new smooth pfaffian Calabi-Yau threefolds in weighted projective spaces, there is an interesting example X_9 .

Definition 5.1. *Set a weighted projective space $\mathbb{P}_{\mathbf{w}_9} = \mathbb{P}_{(1^6, 2)}$. Define the degree 9 Calabi-Yau threefold X_9 as a pfaffian variety associated to the locally free sheaf $\mathcal{E}_9 = \mathcal{O}_{\mathbb{P}_{\mathbf{w}_9}}(2) \oplus \mathcal{O}_{\mathbb{P}_{\mathbf{w}_9}}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}_{\mathbf{w}_9}}^{\oplus 2}$.*

X_9 turns out to be isomorphic to \mathbb{P}_{32}^5 . Therefore X_9 twofold interpretation. If we regard X_9 as a pfaffian Calabi-Yau threefold, we can apply the mirror construction we did to the previous pfaffian cases.

Definition 5.2. Define $\check{\mathcal{X}}_9 = \{\check{X}_{9,t}\}_{t \in \mathbb{P}^1}$ as the one-parameter family of degree 9 pfaffian Calabi-Yau threefolds $\check{X}_{9,t}$ associated to the following special skew-symmetric 5×5 matrix $N_{9,t}$ parametrized by $t \in \mathbb{P}^1$.

$$N_{9,t} = \begin{pmatrix} 0 & x_0x_1x_2 & 0 & tx_6 & x_3x_4 \\ -x_0x_1 & 0 & x_6 & t(x_3+x_4) & tx_5 \\ 0 & -x_6 & 0 & x_5 & t(x_0+x_1+x_2) \\ -tx_6 & -t(x_3+x_4) & -x_5 & 0 & 1 \\ x_3x_4 & -tx_5 & -(x_0+x_1+x_2) & -1 & 0 \end{pmatrix}$$

Indeed X_9 is globally defined by complete intersection of P_0, P_1 and P_2 . The period integral and the Picard-Fuchs operator \mathcal{D}_9 of the conjectural mirror family $\check{\mathcal{X}}_9$ coincides with well-known one

$$\Phi_0(\phi) = \sum_{n=0}^{\infty} \binom{3n}{n}^2 \binom{2n}{n}^2 \phi^n, \quad \mathcal{D}_9 = \Theta^4 - 3^2 \phi (3\Theta + 1)^2 (3\Theta + 2)^2,$$

where we put $\phi = t^8$. Interesting observation is that this one-parameter family $\check{\mathcal{X}}_9$ is not isomorphic to the conventional mirror family for $\mathbb{P}_{3^2}^5$ defined by

$$\begin{aligned} & x_0x_1x_2 + t(x_3^3 + x_4^3 + x_5^3) \\ & x_3x_4x_5 + t(x_0^3 + x_1^3 + x_2^3). \end{aligned}$$

This is a family of smooth Calabi-Yau threefolds, while a general member of $\check{\mathcal{X}}_9$ has singularities along a curve. These two families may bridge two mirror constructions.

It is classically known that a mirror family for a given family Calabi-Yau threefolds can be constructed by taking special loci of the initial family, which are not necessarily on the Fermat points emphasized by the initial construction inspired by the conformal field theories. For more details, see [7], and the reference therein.

APPENDIX

X_5		
d	n_d^0	n_d^1
1	2220	0
2	285520	460
3	95254820	873240
4	47164553340	1498922677
5	28906372957040	2306959237408

X_7		
d	n_d^0	n_d^1
1	1434	0
2	103026	26
3	18676572	53076
4	4988009280	65171063
5	1646787631350	63899034076

X_{10}		
d	n_d^0	n_d^1
1	888	0
2	33084	1
3	3003816	2496
4	399931068	2089393
5	65736977760	1210006912

\tilde{d}	\tilde{n}_d^0	\tilde{n}_d^1
1	$2400a$	40
2	$1829880a$	138040
3	$2956977632a$	687719624
4	$7117422755016a$	3822563543952
5	$21319886408804640a$	21893828822263288

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⁶ a is expected to be 2 in [17].

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